much as possible. Check your answers by multiplying out your factorization.

79. $f(x) = x^3 - 2x^2 - 5x + 6$

80. $f(x) = 3x^3 - 8x^2 + 5x - 2$

81. $f(x) = x^3 + 4x^2 - 11x + 6$

82. $f(x) = x^4 - 8x^3 + 24x^2 - 32x + 16$

83. $f(x) = 3x^4 + 11x^3 + 15x^2 + 9x + 2$

84. $f(x) = 2x^4 + 6x^2 - 8$

85. $f(x) = 3x^4 + 7x^3 - 21x^2 - 6x - 8$

86. $f(x) = 2x^6 - 2x^5 - 10x^4 + 2x^3 + 16x^2 + 8x$

■ **For each polynomial $f(x)$ and linear expression $l(x) = x - c$, (a) use synthetic division to find $\frac{f(x)}{l(x)}$. Then (b) use your answer to write $f(x)$ in terms of $x - c$ and the "remainder" (see Algorithm 5.1). Check your answers by multiplying out this expression for $f(x)$.**

87. $f(x) = 2x^3 + 2x^2 - x - 5, \quad l(x) = x - 3$

88. $f(x) = 2x^3 - 3x^2 + 8, \quad l(x) = x - 2$

89. $f(x) = 1 - x^4, \quad l(x) = x + 3$

90. $f(x) = 5x^4 - 3x + 2, \quad l(x) = x + 1$

91. $f(x) = (x - 2)(x^5 - 3x^2) + 4, \quad l(x) = x - 2$

92. $f(x) = x^6 + x^5 - 3x^3 - 2x^2 - 7x - 8, \quad l(x) = x + 1$

■ **There is an analogue of the Integer Root Theorem that finds *rational* roots of polynomials (sensibly called the *Rational Root Theorem*):**

(1) *If $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where each a_i is an integer and $a_0 \neq 0$, then every (reduced) rational root $\frac{p}{q}$ has the property that p is a positive or negative divisor of the constant term a_0, and q is a positive or negative divisor of the leading coefficient a_n.*

Use this more general theorem to guess the rational roots of each of the following polynomials, and then use synthetic division (which works for dividing a polynomial expression by $x - c$ even if c is not an integer) to factor these polynomials completely.

93. $f(x) = 2x^3 + x^2 + 8x + 4$

94. $f(x) = 6x^4 - 7x^3 + 8x^2 - 7x + 2$

95. $f(x) = 16x^3 - 12x^2 + 1$

96. $f(x) = 27x^4 - 18x^2 - 8x - 1$

Proofs

97. Prove the two implications of the Factor Theorem: If f is a polynomial function, then **(a)** if $x - c$ is a factor of f, then $x = c$ is a root of f, and **(b)** if $x = c$ is a root of f, then $x - c$ is a factor of f.

98. Suppose f is a polynomial function that factors as $f(x) = 3(x - a)(x^2 + cx + d)$, where $c^2 < 4d$. Prove that f has only one real root.

99. Prove that if f is a polynomial function whose constant term is $a_0 = 0$, then $x = 0$ is a root of f.

5.2 Limits and Derivatives of Polynomial Functions

In this section we will show that polynomial functions are always continuous, differentiable, and defined for all real numbers. Every property of polynomial functions will follow directly from the fact that polynomials are sums of power functions with positive integer powers.

5.2.1 Continuity of polynomial functions

We begin with a simple theorem that describes the domain and continuity of polynomial functions.

THEOREM 5.5

Continuity of Polynomial Functions

Every polynomial function is continuous and defined on all of $(-\infty, \infty)$.

This theorem is easy to prove using the fact that polynomials are sums of power functions with integer powers.

PROOF **(THEOREM 5.5)** Let f be a polynomial function of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0.$$

By definition, f is a sum of power functions whose powers are nonnegative integers (namely, the power functions $a_n x^n$, $a_{n-1}x^{n-1}$, ..., $a_1 x$, and a_0). From Chapter 4 we know that power functions with nonnegative integer powers are defined for all real numbers. Since f is a sum of such power functions, it is also defined for all real numbers.

We know from Section 4.2.1 that every power function is continuous (on its entire domain). Moreover, we know that sums of continuous functions are themselves continuous functions. Therefore, f is a sum of continuous functions and is thus continuous everywhere on its domain $(-\infty, \infty)$. ■

Recall that a function f is continuous at a point $x = c$ if and only if its limit as $x \to c$ is its value $f(c)$. We can always calculate limits of polynomial functions by evaluating at $x = c$ (since every polynomial function is continuous). If n is any positive integer and $a_0, a_1, \ldots, a_n$ are any real numbers, then for all real numbers c,

$$\lim_{x\to c}(a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0) = a_n c^n + a_{n-1}c^{n-1} + \cdots + a_1 c + a_0.$$

For example, $\lim\limits_{x\to 3}(2x^4 - 5x^2 - 8) = 2(3)^4 - 5(3)^2 - 8 = 109$. In general, if f is a polynomial function, then for any real number c, we have $\lim\limits_{x\to c} f(x) = f(c)$.

5.2.2 Global behavior of polynomial functions

Since every polynomial function is continuous on $(-\infty, \infty)$, we know how to compute $\lim\limits_{x\to c} f(x)$ for any polynomial function $f(x)$ and real number c. We now focus on the limits of polynomial functions as $x \to \infty$ and as $x \to -\infty$. These limits will determine part of the ***global behavior*** of polynomial functions, namely how polynomial functions behave at their "ends."

As the following theorem shows, the limit of a polynomial function f as $x \to \infty$ or as $x \to -\infty$ depends only on the sign of the leading coefficient and the parity of the degree of f.

THEOREM 5.6

Global Behavior of Polynomial Functions

Suppose $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is a polynomial function of degree n. If the leading coefficient a_n is positive, then

(a) If n is even, then $\lim\limits_{x\to\infty} f(x) = \infty$ and $\lim\limits_{x\to-\infty} f(x) = \infty$.

(b) If n is odd, then $\lim\limits_{x\to\infty} f(x) = \infty$ and $\lim\limits_{x\to-\infty} f(x) = -\infty$.

If the leading coefficient a_n is negative, then the signs above are reversed.

Figures 5.9–5.12 illustrate Theorem 5.6 graphically. The dashed part in the middle of each graph indicates that this theorem does not tell us the behavior of the graph in the "middle," only at the "ends" (the dashed part is just an example of what *might* happen in the middle part of the graph).

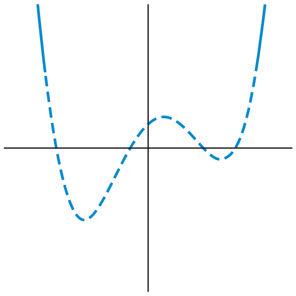


Figure 5.9
n even, $a_n > 0$

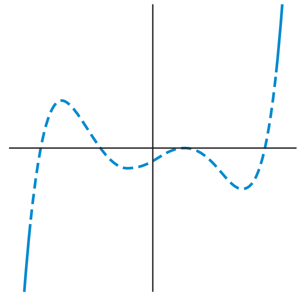


Figure 5.10
n odd, $a_n > 0$

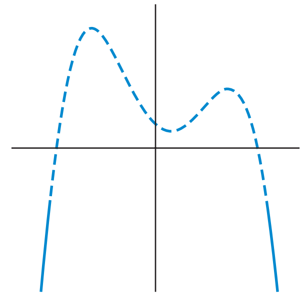


Figure 5.11
n even, $a_n < 0$

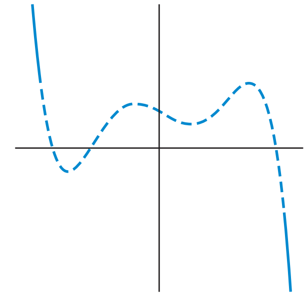


Figure 5.12
n odd, $a_n < 0$

The "ends" of the graph of an even-degree polynomial either both point up or both point down, whereas the "ends" of the graph of an odd-degree polynomial point in opposite directions. In particular, a polynomial function always increases or decreases without bound as $x \to \infty$ and as $x \to -\infty$; therefore, the graph of a polynomial function never has a horizontal asymptote.

EXAMPLE 5.11 **Limits at the "ends" of a polynomial function**

The polynomial function $f(x) = -3x^5 + 3x^2 + 10$ has leading coefficient $a_n = -3$ and degree $n = 5$. Since the degree is odd, the "ends" of the graph of f will point in opposite directions. Since the leading coefficient is negative, the height of the graph will approach infinity ("go up") at the left end and negative infinity ("go down") at the right end. In the language of limits,

$$\lim_{x \to -\infty} (-3x^5 + 3x^2 + 10) = \infty \quad \text{and} \quad \lim_{x \to \infty} (-3x^5 + 3x^2 + 10) = -\infty.$$

Figure 5.13 shows the graph of $f(x) = -3x^5 + 3x^2 + 10$. Verify that this graph does indeed have the global behavior predicted by Theorem 5.6.

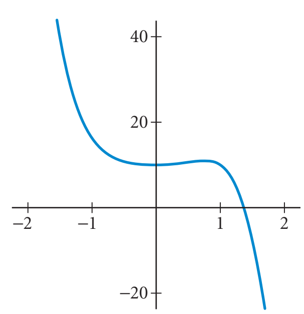


Figure 5.13
$f(x) = -3x^5 + 3x^2 + 10$

□

The behavior at the "ends" of a polynomial function f depends entirely on the leading term of that polynomial (the remaining terms determine the local behavior of f, as we will see in the next section). This is because for values of x with very large (positive or negative) magnitude, the leading term of a polynomial f is the "most powerful." In fact, as $x \to \infty$ and as $x \to -\infty$, a polynomial will behave more and more like its leading term. This will be the main argument in our proof of Theorem 5.6. Before we get to the proof, let's look at an example that illustrates the "power" of the leading term of a polynomial.

EXAMPLE 5.12 **The global behavior of a polynomial is determined by its leading term**

Consider the polynomial $f(x) = x^4 - x^3 - 11x^2 + 9x + 18$. The shape of the graph of f at the "ends" is determined by the leading term, x^4. Figures 5.14–5.16 show graphs of the function $y = x^4 - x^3 - 11x^2 + 9x + 18$ and its leading term $y = x^4$ (the dashed curve) in different viewing windows. The more we zoom out (i.e., as $x \to \infty$ or as $x \to -\infty$), the more the graph of the function f looks like the graph of $y = x^4$.

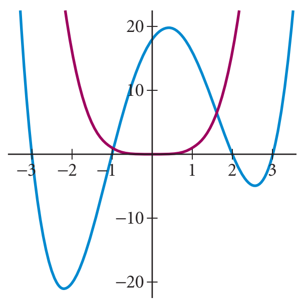


Figure 5.14
$y = x^4 - x^3 - 11x^2 + 9x + 18$ and $y = x^4$ on $[-3, 3]$

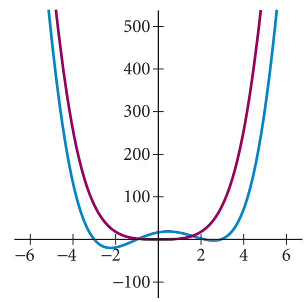


Figure 5.15
$y = x^4 - x^3 - 11x^2 + 9x + 18$ and $y = x^4$ on $[-6, 6]$

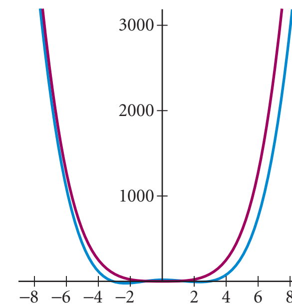


Figure 5.16
$y = x^4 - x^3 - 11x^2 + 9x + 18$ and $y = x^4$ on $[-8, 8]$

□

Why would the limit of a polynomial as $x \to \infty$ or as $x \to -\infty$ be difficult to compute? After all, a polynomial is just a sum of very simple power functions! The problem arises when we are forced to add or subtract "infinities"; for example, what is the limit as $x \to \infty$ of the function $f(x) = x^3 - 12x^2$? We know that as $x \to \infty$ we have $x^3 \to \infty$ and $-12x^2 \to -\infty$. However, we cannot simply add these two limits together to find the limit of $f(x)$, since limits of the form $\infty - \infty$ are indeterminate.

? ***Question*** Find the limits of $f(x) = x^2 - 2x^2$, $g(x) = 2x^2 - x^2$, and $h(x) = x^2 - x^2$ as $x \to \infty$ (each of these limits starts out in the form "$\infty - \infty$"; simplify first to find the limit). Use the results to show that limits of the form "$\infty - \infty$" are indeterminate. □

We will now prove Theorem 5.6 by showing that the limit of a polynomial function as $x \to \infty$ or as $x \to -\infty$ is determined entirely by the limit of its leading term.

PROOF (THEOREM 5.6) Suppose $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is any polynomial function. We will first show that the limit of this polynomial function as $x \to \infty$ is equal to the limit of its leading term as $x \to \infty$:

$$\lim_{x\to\infty} (a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0) = \lim_{x\to\infty} (a_n x^n).$$

This is just a simple calculation; the key is to factor out the leading term:

$$\lim_{x\to\infty} (a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0)$$

$$= \lim_{x\to\infty} \left(a_n x^n \left(1 + \frac{a_{n-1}x^{n-1}}{a_n x^n} + \cdots + \frac{a_1 x}{a_n x^n} + \frac{a_0}{a_n x^n} \right) \right) \quad \textbf{(factor out leading term)}$$

$$= \lim_{x\to\infty} \left(a_n x^n \left(1 + \frac{a_{n-1}}{a_n x} + \cdots + \frac{a_1}{a_n x^{n-1}} + \frac{a_0}{a_n x^n} \right) \right) \quad \textbf{(algebra)}$$

$$= \left(\lim_{x\to\infty} a_n x^n \right) \left(\lim_{x\to\infty} \left(1 + \frac{a_{n-1}}{a_n x} + \cdots + \frac{a_1}{a_n x^{n-1}} + \frac{a_0}{a_n x^n} \right) \right). \quad \textbf{(limit rule)}$$

Each of the quotients in the second limit approaches zero as $x \to \infty$ (why?), and thus the second limit is equal to 1. Continuing our calculation, the original limit is equal to

$$\left(\lim_{x\to\infty} a_n x^n\right)(1) = \lim_{x\to\infty} a_n x^n.$$

Technical Note: We applied the product rule for limits in this calculation, which we have shown to be valid only when all limits involved exist. In this example, some limits are infinite (and thus "do not exist"), so technically our product rule for limits does not apply. However, we *can* use the product rule for limits even when the limits involved "do not exist" if we are careful with how we handle "infinity" and indeterminate forms, although we will not prove this here.

We now continue with the proof of Theorem 5.6. We have shown that the limit at infinity of any polynomial function is equal to the limit at infinity of its leading term. Since that leading term is a power function (with a positive integer power), we can use what we know about the limits of power functions at infinity to find the limits of polynomial functions at infinity. We know that if n is an even or odd positive integer, then $\lim_{x\to\infty} x^n = \infty$. Therefore, we have

$$\lim_{x\to\infty} a_n x^n = \infty \text{ if } a_n > 0, \quad \text{and} \quad \lim_{x\to\infty} a_n x^n = -\infty \text{ if } a_n < 0.$$

By our previous calculation, this means that whether n is even or odd,

$$\lim_{x\to\infty} (a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0) = \lim_{x\to\infty} a_n x^n = \infty \text{ if } a_n > 0, \quad \text{and}$$

$$\lim_{x\to\infty} (a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0) = \lim_{x\to\infty} a_n x^n = -\infty \text{ if } a_n < 0.$$

The proof for the limit of a polynomial function as $x \to -\infty$ is similar; it is left to the exercises. The major difference will be in the last couple of steps, where it will be important whether n is even or odd, since we know that if n is a positive even integer, then $\lim_{x\to-\infty} x^n = \infty$, whereas if n is a positive odd integer, then $\lim_{x\to-\infty} x^n = -\infty$. ■

Caution The statement that a polynomial and its leading term have the same limit applies only as $x \to \infty$ or as $x \to -\infty$. A polynomial and its leading term usually have different limits as $x \to c$. For example, $\lim_{x\to 2}(2x^3 - 3x + 1) = 16 - 6 + 1 = 11$, whereas $\lim_{x\to 2}(2x^3) = 16$. □

5.2.3 Derivatives of polynomial functions

At this point we already know how to differentiate a polynomial function using the power rule, sum rule, and constant multiple rule, since every polynomial function is a sum of constant multiples of power functions x^k. In general, the derivative of a polynomial is given by the following formula.

THEOREM 5.7

Differentiating Polynomial Functions

If $f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2 x^2 + a_1 x + a_0$ is any polynomial function, then

$$f'(x) = na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + (n-2)a_{n-2}x^{n-3} + \cdots + 2a_2 x + a_1.$$

You don't need to memorize this formula; you just need to be able to use the power, sum, and constant multiple rules to differentiate any polynomial function quickly. For example, $\frac{d}{dx}(2x^5 - 5x^3 - 4x + 12) = 2(5x^4) - 5(3x^2) - 4(1) + 0 = 10x^4 - 15x^2 - 4$. You will use these same differentiation rules to prove Theorem 5.7 in the exercises.

Since every polynomial function has a well-defined derivative (as given in Theorem 5.7) at every point in its domain $(-\infty, \infty)$, polynomial functions are always differentiable.

THEOREM 5.8

Differentiability of Polynomial Functions

Every polynomial function is differentiable on all of $(-\infty, \infty)$.

Since polynomials are defined, continuous, and differentiable for all real numbers, their graphs are unbroken and smooth everywhere. We will use these nice properties to identify and graph polynomial functions in Section 5.3.

The derivative of a polynomial function is itself a polynomial function. One might then expect that the *anti*derivative of a polynomial function would itself be a polynomial function, and this is in fact the case. Antiderivatives of polynomial functions can be calculated by using the sum and constant multiple rules and the antiderivatives of power functions.

EXAMPLE 5.13

Finding an antiderivative of a polynomial function

Find a function $f(x)$ whose derivative is $f'(x) = 5x^4 - 2x^3 + 3x - 4$ that has value $f(0) = 3$.

Solution From Section 4.3.6 we know that for every positive integer k, if $f'(x) = x^k$ then $f(x) = \frac{1}{k+1}x^{k+1} + C$ for some positive integer C. Using this rule (and the sum and constant multiple rules), we thus calculate that every antiderivative of $f'(x) = 5x^4 - 2x^3 + 3x - 4$ has the form

$$\begin{aligned} f(x) &= 5\left(\tfrac{1}{5}x^5\right) - 2\left(\tfrac{1}{4}x^4\right) + 3\left(\tfrac{1}{2}x^2\right) - 4\left(\tfrac{1}{1}x^1\right) + C \\ &= x^5 - \tfrac{1}{2}x^4 + \tfrac{3}{2}x^2 - 4x + C. \end{aligned}$$

Given that $f(0) = 3$, we can solve for C:

$$f(0) = 3 \implies 0^5 - \tfrac{1}{2}0^4 + \tfrac{3}{2}0^2 - 4(0) + C = 3 \implies C = 3.$$

Therefore, we must have $f(x) = x^5 - \frac{1}{2}x^4 + \frac{3}{2}x^2 - 4x + 3$. □

✓ ***Checking the Answer*** To check the answer, verify that the function $f(x) = x^5 - \frac{1}{2}x^4 + \frac{3}{2}x^2 - 4x + 3$ has derivative $f'(x) = 5x^4 - 2x^3 + 3x - 4$ and value $f(0) = 3$. □

Recall that the position, velocity, and acceleration functions ($s(t)$, $v(t)$, and $a(t)$, respectively) of a moving object are related by the equations

$$v(t) = s'(t), \qquad a(t) = v'(t), \qquad a(t) = s''(t).$$

If any one of these three functions is a polynomial, then we can find the other two functions by differentiating or antidifferentiating, as appropriate. For example, if $v(t)$ is a given polynomial, then we can differentiate $v(t)$ to get $a(t) = v'(t)$ and antidifferentiate $v(t)$ to get $s(t)$ (since $v(t) = s'(t)$). Remember that the initial conditions $s_0 = s(0)$, $v_0 = v(0)$, and $a_0 = a(0)$ represent the initial position, velocity, and acceleration of the moving object, respectively. These conditions can be used to solve for the constant "C" that arises in any antidifferentiation calculations.

EXAMPLE 5.14 **Obtaining position and acceleration functions from velocity**

A spaceship is moving along a straight path from Venus straight into the heart of the sun. The velocity of the spaceship t hours after leaving Venus is $v(t) = 0.015t^2 + 200$ thousands of miles per hour. Find equations for the position and acceleration of the spaceship t hours after leaving Venus. What can you say about the initial values s_0, v_0, and a_0?

Solution From the equation $v(t) = 0.015t^2 + 200$, we can see that the initial velocity of the spaceship is $v_0 = v(0) = 0.015(0)^2 + 200 = 200$ thousand miles per hour.

Since $a(t) = v'(t)$, we know that

$$a(t) = \frac{d}{dt}(0.015t^2 + 200) = 2(0.015)t + 0 = 0.03t.$$

In particular, the initial acceleration of the spaceship is $a_0 = a(0) = 0.03(0) = 0$ thousands of miles per hour per hour.

Finally, since $v(t) = s'(t)$, we know that $s'(t) = 0.015t^2 + 200$. Therefore, we can antidifferentiate $0.015t^2 + 200$ to get a formula for $s(t)$:

$$s(t) = 0.015\left(\tfrac{1}{3}t^3\right) + 200(t) + C = 0.005t^3 + 200t + C.$$

By assumption, the spaceship starts out at Venus at position $s_0 = 0$; we can use this initial position to solve for C:

$$s(0) = 0 \implies 0.005(0)^3 + 200(0) + C = 0 \implies C = 0.$$

Therefore, t hours after leaving, the spaceship is $s(t) = 0.005t^3 + 200t$ thousand miles from Venus. □

5.2.4 Local behavior of polynomial functions

The global behavior of a polynomial function is determined solely by its leading term. On the other hand, *all* of the terms of a polynomial function help determine the ***local behavior*** of the function. Examples of the local behavior of a function include roots, maxima, minima, and inflection points.

Recall that finding the extrema and inflection points of a function involves determining the zeros and signs of the first and second derivatives of that function. Since the derivatives of polynomial functions are always themselves polynomial functions, much of the local behavior of a polynomial function f can be described by examining roots and signs of polynomial functions (namely, f, f', and f''). Here is where all our factoring skills from Section 5.1 will come in handy!

EXAMPLE 5.15 **Using derivatives to determine the local behavior of a polynomial function**

Find all the local extrema and inflection points of the function $f(x) = x^4 + 4x^3 - 16x$.

Solution To find the local extrema of f, we must first find its critical points. The derivative of f is $f'(x) = 4x^3 + 12x^2 - 16$, which is a polynomial and thus is always defined; we need only find its zeros. First we pull out the common factor of 4:

$$f'(x) = 4x^3 + 12x^2 - 16 = 4(x^3 + 3x^2 - 4).$$

The polynomial $x^3 + 3x^2 - 4$ looks difficult to factor at first glance (no grouping or common factors), so we'll guess some roots. The constant term is -4, so by the Integer Root Theorem, the only possible integer roots are ±1, ±2, and ±4. We only need to find one root to use synthetic division and reduce the problem of factoring a cubic to a problem of factoring a quadratic. We test the easiest root possibility, $x = 1$, and find that it is in fact a root, since $(1)^3+3(1)^2-4 = 1+3-4 = 0$. Now a quick synthetic-division calculation (see Figure 5.17) will enable us to factor $x - 1$ from $x^3 + 3x^2 - 4$.

$$\begin{array}{r|rrrr} & 1 & 3 & 0 & -4 \\ 1 & & 1 & 4 & 4 \\ \hline & 1 & 4 & 4 & 0 \end{array}$$

Figure 5.17

Thus $x^3 + 3x^2 - 4 = (x - 1)(x^2 + 4x + 4)$ (you can check this by multiplying out), so

$$\begin{aligned} f'(x) &= 4x^3 + 12x^2 - 16 \\ &= 4(x^3 + 3x^2 - 4) \\ &= 4(x - 1)(x^2 + 4x + 4) && \textbf{(synthetic division)} \\ &= 4(x - 1)(x + 2)(x + 2). && \textbf{(factoring a quadratic)} \end{aligned}$$

The only zeros of f' (and thus the only critical points of f) are $x = 1$ and $x = -2$.

To find the local extrema of f, we test each of these critical points to determine whether it is a local maximum, a local minimum, or neither. A sign analysis of f' gives us the number line in Figure 5.18 (check this by evaluating $f'(-3)$, $f'(0)$, and $f'(2)$).

Figure 5.18

Since the sign of f' changes only at $x = 1$ (and not at $x = -2$), the only local extremum of f is at $x = 1$. The sign of f' changes from negative to positive at $x = 1$, so by the first derivative test, f has a local minimum at $x = 1$.

To find the inflection points of $f(x) = x^4 + 4x^3 - 16x$, we must do a similar analysis for the second derivative of f. The second derivative is the derivative of $f'(x) = 4x^3 + 12x^2 - 16$, which is $f''(x) = 12x^2 + 24x$. Luckily this is very easy to factor:

$$f''(x) = 12x^2 + 24x = 12x(x + 2).$$

The only possible inflection points of f are at $x = 0$ and $x = -2$. By evaluating, for example, $f''(-3)$, $f''(-1)$, and $f''(1)$, we can quickly get a sign analysis for f'' (see Figure 5.19).

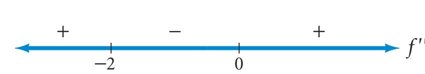


Figure 5.19

Since the sign of f'' changes both at $x = -2$ and at $x = 0$, the function f changes concavity at both of these values. The function f has two inflection points, at $x = -2$ and at $x = 0$. □

✓ ***Checking the Answer*** The graph of the function $f(x) = x^4+4x^3-16x$ is shown in Figure 5.20. Note that $x = -2$ is a critical point that is not a local extremum, and $x = 1$ is a critical point that is a local minimum. Moreover, both $x = -2$ and $x = 0$ are inflection points of f.

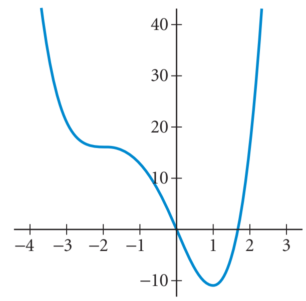


Figure 5.20
$f(x) = x^4 + 4x^3 - 16x$ □

A point $x = c$ is a ***turning point*** of the graph of a function f if the graph "turns around" at that point. More precisely, the turning points of a function f are the local maxima and minima of f. The following theorem tells us how many turning points a polynomial function of degree n can have.

THEOREM 5.9

The Maximum Number of Turning Points for a Polynomial Function

A polynomial function of degree n can have at most $n - 1$ turning points.

Caution Note the words "at most" in Theorem 5.9; a polynomial function of degree n can have $n - 1$ *or fewer* turning points. □

EXAMPLE 5.16

A polynomial of degree 5 has at most four turning points

Consider the three quintic polynomials shown in Figures 5.21–5.23.

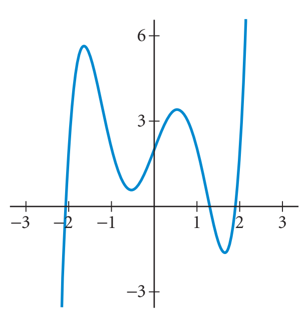


Figure 5.21
$f(x) = x^5 - 5x^3 + 4x + 2$

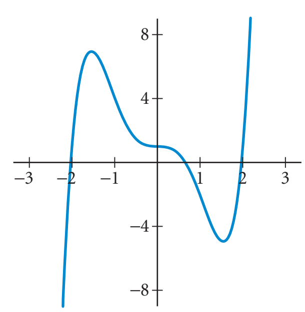


Figure 5.22
$g(x) = x^5 - 4x^3 + 1$

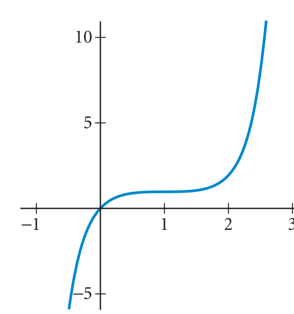


Figure 5.23
$h(x) = x^5 - 5x^4 + 10x^3 - 10x^2 + 5x$

The function f in Figure 5.21 has four turning points (the maximum amount, since $5 - 1 = 4$). The function g has only two turning points, and the function h has no turning points at all. All of these polynomial functions are of degree 5 and have *at most* four turning points. □

The proof of Theorem 5.9 follows directly from the Fundamental Theorem of Algebra, since every turning point of a polynomial function f is a critical point of f and thus a zero of the polynomial function f'.

PROOF (THEOREM 5.9) Suppose f is an nth-degree polynomial function. Then by Theorem 5.7, its derivative f' is a polynomial of degree $n - 1$. Since the polynomial f' is defined for all real numbers (every polynomial is), the only critical points of f are the zeros of f'. By the Fundamental Theorem of Algebra, the derivative f' can have at most $n-1$ roots (since it is of degree $n-1$), and thus the function f can have at most $n-1$ critical points. Moreover, some of these critical points may be turning points of f, and some may not. Thus there are at the very *most* $n-1$ turning points for f. ■

Given the graph of a polynomial function, Theorems 5.6 and 5.9 can help us identify the parity and size of the degree of that polynomial function. This information will be especially useful in the next section when we try to find polynomial functions to model graphs.

? ***Question*** Can an even-degree polynomial have an even number of turning points? Why or why not? What can you say about whether the number of turning points of an odd-degree polynomial must be odd or even? □

What you should know after reading Section 5.2

- How to prove that polynomial functions are defined, continuous, and differentiable on all of $(-\infty, \infty)$.
- How to calculate limits, derivatives, and antiderivatives of polynomial functions (and why these calculations work). Also, be able to use derivatives and antiderivatives to solve problems involving position, velocity, and acceleration.
- Be able to tell from the global behavior of the graph of a polynomial function f whether the degree of f must be odd or even, and be able to say something about the size of the degree of f by looking at the number of turning points in the graph.
- Be able to find any local extrema and inflection points of a polynomial function f. Note that this may involve some fancy polynomial factoring.

Exercises 5.2

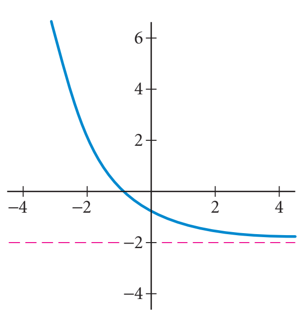
Concepts

0. Read the section and make your own summary of the material. Include whatever you think will help you review the material later on.

1. Fill in the blanks: Every polynomial function is ________, ________, and ________ on $(-\infty, \infty)$.

■ **Explain why each graph cannot possibly be the graph of a polynomial function.**

2.



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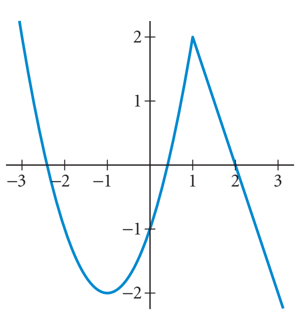

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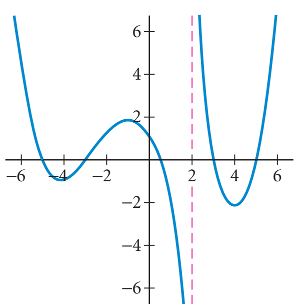
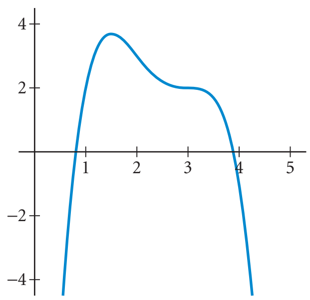


■ **Given the polynomial functions graphed below, what can you say about the parity and size of the degree of f, and why? What can you say about the leading coefficient of f, and why?**

5.



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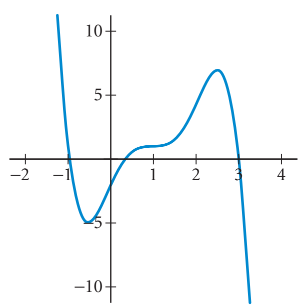
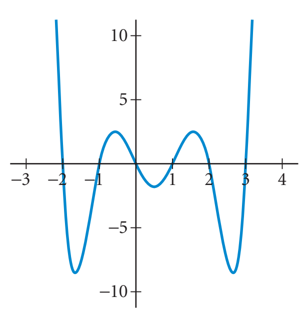


7.



■ **For each problem, either sketch a graph representing a polynomial function with the given conditions, or explain why such a polynomial function cannot exist.**

8. f is a polynomial function of odd degree and negative leading coefficient.

9. f is a polynomial function of degree 4 with five turning points.

10. f is a polynomial function of even degree, three turning points, and one root.

11. f is a polynomial function of degree 5 with three turning points.

12. f is a polynomial function of even degree that is never positive.

13. f is a polynomial function of degree 7 that has seven roots and four turning points.

14. f is a polynomial function of degree 6 that has five turning points and three roots.

15. Use a graphing calculator to sketch the graphs of $f(x) = x^5 - 2x^4 - 3x^3 + 8x^2 - 4x$ and $g(x) = x^5$ in two different graphing windows: first, a relatively small window where the local behavior of these two functions is clear (the two graphs should look very different); second, a graphing window large enough that the graphs look almost identical. Why is it possible to find a window large enough so that the two graphs look almost the same?

16. Consider the functions $f(x) = x^2 - 2x^2$, $g(x) = 2x^2 - x^2$, and $h(x) = x^2 - x^2$. The limit of each of these functions as $x \to \infty$ is initially of the form $\infty - \infty$. Simplify these functions so that you can compute their limits as $x \to \infty$, and use your answers to show that limits of the form $\infty - \infty$ are indeterminate.

17. Can an even-degree polynomial have an even number of turning points? An odd number? Why or why not?

18. Can an odd-degree polynomial have an even number of turning points? An odd number? Why or why not?

■ **Determine whether each of the following statements is true or false. If a statement is true, explain why. If a statement is false, provide a counterexample.**

19. True or False: If f is a function whose graph has a horizontal asymptote, then f cannot be a polynomial function.

20. True or False: If f is a polynomial function of degree n, then the graph of f must have $n - 1$ turning points.

21. True or False: If f is a polynomial function with k roots, then f must have at least $k - 1$ turning points.

22. True or False: If f is a polynomial function of odd degree, then f must have at least one real root.

23. True or False: If f is a polynomial function with leading term $a_n x^n$, then $\lim_{x\to 3} f(x) = \lim_{x\to 3} a_n x^n$.

24. True or False: If f is a polynomial function with leading term $a_n x^n$, then $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} a_n x^n$.

25. True or False: If f is a polynomial function of even degree, then f is an even function.

26. True or False: If f is a polynomial function of odd degree, then f is an odd function.

Skills

■ **Calculate the following limits.**

27. $\lim_{x\to -1} (3 - 5x^3 + 2x^7)$

28. $\lim_{x\to 3} (2x^4 - 12x^2 + 2)$

29. $\lim_{x\to 0} (2x^6 - 5)(3x^4 - 2x^3)$

30. $\lim_{x\to\infty} (-5x^4 + 3x^2 - 10)$

31. $\lim_{x\to\infty} (-2x^5 + 8x^4 - 6)$

32. $\lim_{x\to -\infty} (2x^7 + x^4 - x^3 + 16)$

33. $\lim_{x \to -\infty} (3x^3 - 1)(5 - 7x^4)$

34. $\lim_{x \to \infty} \frac{1}{1 - 2x^7}$

35. $\lim_{x \to -\infty} \frac{-2}{3x^5 - 2x^2 + 17}$

36. $\lim_{x \to -2} \frac{x^2 - 4x + 4}{x^5 - 6x^4 + 12x^3 - 8x^2}$

37. $\lim_{x \to 3} \frac{x^4 - 4x^3 + 2x^2 + 4x - 3}{x^2 - 5x + 6}$

38. $\lim_{x \to -\infty} \frac{x^3 - 3x + 2}{x^3 + 2x^2 - x - 2}$

■ **Calculate the following derivatives using the sum, constant multiple, and power rules. Some algebra may be needed before differentiating.**

39. $\frac{d}{dx}(5x^6 - 3x^4 + 2x^3 - 3x + 6)$

40. $\frac{d}{dx}(2 - 3x^3 + x^8 - 4x^{11})$

41. $\frac{d^2}{dx^2}(3x^5 - 2x^4 + 3x^2 - 5x + 7)$

42. $\frac{d}{dx}(3x + 1)(4 - x^7)$

43. $\frac{d}{dx}((2x - 1)^4)$

44. $\frac{d^3}{dx^3}(3x^5 + 5(1 - 2x^4))$

45. $\frac{d^5}{dx^5}(3x^5 - 2x^4 + 7x^2 - 1)$

46. $\frac{d^{11}}{dx^{11}}(-2x^8 + 3x^5 - 2x^2 + 6)$

47. $\frac{d^7}{dx^7}(4x^7 + 3x^2 - 5)$

■ **For each problem, find a function f that has the given derivative and value. Check your answer by checking that your function f does indeed have the given derivative and value.**

48. $f'(x) = 3x^5 - 2x^2 + 4, \quad f(0) = 1$

49. $f'(x) = 7x^2 + 8x^{11} - 18, \quad f(0) = -2$

50. $f'(x) = 1 - 4x^6, \quad f(1) = 3$

51. $f'(x) = x(4 - 2x), \quad f(0) = 0$

52. $f'(x) = (3x + 1)^3, \quad f(2) = 1$

53. $f'(x) = (x^4 - 8)(1 - 3x^5), \quad f(0) = 2$

■ **Find all local extrema and inflection points of each polynomial function f. Do all calculations by hand, and show and explain your work carefully. Check your work with a graphing calculator when you are finished.**

54. $f(x) = 2x^3 - 3x^2 - 12x$

55. $f(x) = x^4 - 6x^2 + 8x$

56. $f(x) = x^{11} + 4$

57. $f(x) = x^4 - 4x^3 + 6x^2 - 36x$

58. $f(x) = x^4 + 4x^3 + 6x^2 + 8x$

59. $f(x) = 2x^5 - 15x^4 - 30x^3 + 70x^2$

60. $f(x) = x^4 - 2x^3 - 8x^2 - 6x$

61. $f(x) = x^3(1 + 3x)$

62. $f(x) = (x^2 - 2x)(x^2 - 2x - 6)$

Applications

■ **In each of the next three problems, use the information given to find functions and initial values for position, velocity, and acceleration. In other words, find equations for $s(t)$, $v(t)$, and $a(t)$, and find the values of s_0, v_0, and a_0.**

63. A mouse is running back and forth on a straight track. He runs along the track for 6 minutes, and his position after t minutes is $s(t) = 0.1t^4 + 1.15t^3 - 4.1t^2 + 4.4t + 3$ feet from the left end of the track.

64. Alina is driving from New York to San Francisco. At 12:00 noon she is 550 miles into her trip. Her velocity t hours after noon is $v(t) = 40t^3 - 100t^2 + 50t + 45$ miles per hour.

65. On the planet XV-37, gravity acts differently than it does here on Earth. An object dropped from a 1000-foot building on planet XV-37 will have a downward gravitational acceleration of $a(t) = -6t$ feet per second per second after falling for t seconds.

66. A spaceship is moving along a straight path from Venus straight into the heart of the sun. The velocity of the spaceship t hours after leaving Venus is $v(t) = 0.012t^2 + 400$ thousands of miles per hour.

   (a) Find equations for the position and acceleration of the spaceship (make sure to include the appropriate units). Say what you can about the initial values s_0, v_0, and a_0.

   (b) Is the spaceship always moving toward the sun? How can you tell?

   (c) Is the spaceship traveling at a constant acceleration? If not, is it *always* speeding up or slowing down as it approaches the sun? How can you tell?

   (d) How long will it take the spaceship to reach the sun? How fast will it be going when it gets there? (Assume that the distance between Venus and the sun is 67 million miles.)

67. After you drink a cup of coffee, the concentration of caffeine in your bloodstream increases at first and then

decreases steadily. Let $C(t)$ represent the amount of caffeine (in grams) in your bloodstream t hours after you drink a cup of coffee. Explain why $C(t)$ cannot possibly be a polynomial function. (*Hint:* Try drawing a possible graph of $C(t)$.)

68. On Earth, a falling object has a downward acceleration of -32 feet per second per second because of gravity. Suppose an object falls from an initial height of s_0 feet, with an initial velocity of v_0 feet per second. Show that the equations for the position and velocity of the falling object after t seconds are

$$s(t) = -16t^2 + v_0 t + s_0 \quad \text{and} \quad v(t) = -32t + v_0.$$

Proofs

69. Prove in your own words that every polynomial function is defined on all of $(-\infty, \infty)$. (*Hint:* Use what you know about power functions.)

70. Prove in your own words that every polynomial function is continuous on all of $(-\infty, \infty)$. (*Hint:* Use what you know about power functions.)

71. Prove in your own words that every polynomial function is differentiable on all of $(-\infty, \infty)$. (*Hint:* Use what you know about power functions.)

72. Prove that if $f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2 x^2 + a_1 x + a_0$ is any polynomial function, then its derivative is

$$f'(x) = na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + (n-2)a_{n-2}x^{n-3} + \cdots + 2a_2 x + a_1.$$

73. Suppose $g(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2 x^2 + a_1 x + a_0$ is any polynomial function. Prove that an antiderivative of g is

$$f(x) = \frac{a_n}{n+1}x^{n+1} + \frac{a_{n-1}}{n}x^n + \frac{a_{n-2}}{n-1}x^{n-1} + \cdots + \frac{a_2}{3}x^3 + \frac{a_1}{2}x^2 + a_0 x + C.$$

(*Hint:* It suffices to show that $f'(x) = g(x)$. Why?)

74. Suppose f is a polynomial function of odd degree. Use the global behavior of f and the Intermediate Value Theorem to prove that f must have at least one real root.

75. Suppose f is a polynomial function with k roots. Use Rolle's Theorem to prove that f must have at least $k-1$ turning points.

76. Prove the parts of Theorem 5.6 that were not proved in the text. In other words, prove that if f is a polynomial function of degree n with leading coefficient a_n, then

(a) If n is even and $a_n > 0$, then $\lim_{x \to -\infty} f(x) = \infty$.

(b) If n is even and $a_n < 0$, then $\lim_{x \to -\infty} f(x) = -\infty$.

(c) If n is odd and $a_n > 0$, then $\lim_{x \to -\infty} f(x) = -\infty$.

(d) If n is odd and $a_n < 0$, then $\lim_{x \to -\infty} f(x) = \infty$.

77. Prove that the $(n+1)$st derivative of any nth-degree polynomial function is zero.

78. Prove that the nth derivative of an nth-degree polynomial function with leading coefficient a_n is equal to $n!a_n$. (Recall that $n!$ stands for "n factorial," which is equal to the product $n(n-1)(n-2)\cdots(2)(1)$ of all integers from 1 to n.)

5.3 Graphing Polynomial Functions

In this section we use the information from the previous sections to construct graphs (quick sketches as well as precise, accurate graphs) of polynomial functions. We will also see how to model graphs with polynomial functions.

5.3.1 Repeated roots

As we have seen, every polynomial function can be split into linear and irreducible quadratic factors, and each linear factor corresponds to a root of the function. Sometimes a linear factor appears more than once in the factorization; for example, consider the (factored) polynomial function

$$f(x) = (x+1)(x-2)(x-2)(x-6),$$

or, equivalently,

$$f(x) = (x+1)(x-2)^2(x-6).$$